Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 217 (2008) 1368–1380
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

On the local equatorial characterization of zonoids and intersection bodies ☆

Fedor Nazarov [a], Dmitry Ryabogin [b], Artem Zvavitch [b,*]

[a] *Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA*
[b] *Department of Mathematics, Kent State University, Kent, OH 44242, USA*

Received 30 April 2007; accepted 10 August 2007

Available online 24 October 2007

Communicated by Erwin Lutwak

Abstract

In this paper we show that there is no local equatorial characterization of zonoids in *odd* dimensions. This gives a negative answer to the conjecture posed by W. Weil in 1977 and shows that the local equatorial characterization of zonoids may be given only in even dimensions. In addition we prove a similar result for intersection bodies and show that there is no local characterization of these bodies.
Published by Elsevier Inc.

MSC: primary 52A15, 52A21

Keywords: Convex bodies; Radon transform; Cosine transform; Fourier transform

1. Introduction

A *zonoid* in $\mathbb{R}^n$ is an origin-symmetric convex body that can be approximated (in the Hausdorff metric) by finite Minkowski sums of line segments. It turns out that zonoids appear in many different contexts in convex geometry, physics, optimal control theory, and functional analysis (we refer the reader to [1,3,4,6,11,19–22]). One of the equivalent definitions of zonoids, useful

☆ Authors were partially supported by Kansas State NSF–CBMS grant DMS0532656. The first author was partially supported by the NSF grant DMS0501067. The second author was partially supported by the NSF grants DMS0400789, DMS0652672, the third author was partially supported by the NSF grants DMS0504049, DMS0652684.

* Corresponding author.
E-mail addresses: fedja@math.msu.edu (F. Nazarov), ryabogin@math.kent.edu (D. Ryabogin), zvavitch@math.kent.edu (A. Zvavitch).

0001-8708/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.aim.2007.08.013

in convex geometry, leads to a notion of a projection body. An origin-symmetric convex body L in $\mathbb{R}^n$ is called a *projection body* if there exists another origin-symmetric convex body K such that the support function of L in every direction is equal to the volume of the hyperplane projection of K orthogonal to this direction: for every $\xi \in S^{n-1}$,

$$h_L(\xi) = \mathrm{Vol}_{n-1}\big(K|\xi^{\perp}\big),$$

$\xi^{\perp} = \{y \in \mathbb{R}^n\colon\ \xi \cdot y = 0\}$. The support function $h_L(\xi) = \max_{x\in L} \xi \cdot x$ is equal to the dual norm $\|\xi\|_{L^*}$ where L^* stands for the polar body of L. From the above definition and Cauchy formula (see [13, p. 25]), we immediately derive the following analytic definition, which will be useful for us in this paper: An origin-symmetric convex body $L \subset \mathbb{R}^n$ is a zonoid if and only if

$$h_L(\xi) = Cos\,\mu(\xi) := \int\limits_{S^{n-1}} |\xi \cdot \theta|\, d\mu(\theta)$$

with some even positive measure μ on S^{n-1}. Finally, a functional analytic definition shows that an origin-symmetric convex body $L \subset \mathbb{R}^n$ is a zonoid if and only if it is a polar body to the unit ball of a subspace of L_1.

It is well known that every origin-symmetric convex body in $\mathbb{R}^2$ is a projection body, but this is no longer true in $\mathbb{R}^n$ for $n \geqslant 3$ (see [13,21]). It is an interesting question how to determine if a given convex body is a zonoid or not. It is very reasonable to assume that one can provide a strictly local characterization of zonoids. This question was posed repeatedly (see [21] for the history of the problem), however W. Weil showed [23] that a *local* characterization of zonoids does not exist. In particular, he showed that *there exists an origin-symmetric convex C^{∞} body $K \subset \mathbb{R}^n$, $n \geqslant 3$, that is not a zonoid but has the following property*: *for every $u \in S^{n-1}$ there exists a zonoid Z_u centered at the origin and a neighborhood $U_u \subset S^{n-1}$ of u such that the boundaries of K and Z_u coincide at all points where the exterior unit normal vectors belong to U_u*. Thus, no characterization of zonoids that involves only arbitrarily small neighborhoods of boundary points is possible.

In 1977, W. Weil (see [23]) proposed the following conjecture about *local equatorial* characterization of zonoids. *Let $L \subset \mathbb{R}^n$ be an origin-symmetric convex body and assume that for any equator $\sigma \subset S^{n-1}$, there exists a zonoid Z_σ and a neighborhood E_σ of σ such that the boundaries of L and Z_σ coincide at all points where the exterior unit vector belongs to E_σ*; *then L is a zonoid*. *Affirmative* answers for *even* dimensions were given independently by G. Panina [18] in 1988 and Goodey and Weil [10] in 1993, but the question was left open in odd dimensions. That was a consequence of the fact that the inversion formulas for the cosine transform are not local in odd dimensions.

In this paper we show that the answer to the conjecture in *odd* dimensions is *negative*. We prove that in both cases (for odd and even dimensions) the answer can be obtained as a consequence of the characterization of zonoids in terms of sections of the polar body, given in [14]. In even dimensions the answer follows directly from the geometric inversion formula for the Cosine transform [14]. The odd dimensional case, on the other hand, requires much more tricky and detailed analysis of the behavior of the inverse Cosine transform.

Our main tool is the Fourier analytic inversion formula from [7] (see Eqs. (3), (4) below or [13, p. 60]). It allows to obtain the results for zonoids together with the results about the intersection bodies. The notion of an *intersection body of star body* was introduced by E. Lutwak [17]. K is called the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction: $\forall\xi \in S^{n-1}$,

$$\rho_K(\xi) = \mathrm{Vol}_{n-1}\big(L \cap \xi^{\perp}\big),$$

where $\rho_K(\xi) = \max\{a\colon a\xi \in K\}$ is the radial function of the body K. Passing to polar coordinates in $\xi^{\perp}$, we derive the following analytic definition of an *intersection body of star body*: K is called the intersection body of L if

$$\rho_K(\xi) = \frac{1}{n-1}\mathfrak{R}\rho_L^{n-1}(\xi) := \frac{1}{n-1}\int\limits_{S^{n-1}\cap\xi^{\perp}} \rho_L^{n-1}(\theta)\,d\theta.$$

Here $\mathfrak{R}$ stands for the spherical Radon transform.

A more general class of *intersection bodies* was defined by R. Gardner [5] and G. Zhang [24] as the closure of intersection bodies of star bodies in the radial metric $d(K,L) = \sup_{\xi\in S^{n-1}}|\rho_K(\xi) - \rho_L(\xi)|$. In this paper we will consider only C^{∞} smooth intersection bodies: a body K is an intersection body if there exists an even nonnegative function f on S^{n-1}, such that the radial function of K is a spherical Radon transform $\mathfrak{R}f$ of f. Since we can always define L: $\rho_L^{n-1}(\theta) = (n-1)f(\theta)$, we will not distinguish between *intersection bodies of star bodies* and *intersection bodies*.

We prove that the local equatorial characterization of intersection bodies is not possible in *odd* dimensions. Namely, we show that one can construct *an origin-symmetric convex body $L \subset \mathbb{R}^n$, $n \geqslant 5$ is odd, such that for any equator $\sigma \subset S^{n-1}$, there exists an intersection body I_σ and a neighborhood E_σ of σ such that the boundaries of L and I_σ coincide at all points of E_σ* (*i.e.* $\rho_L(\xi) = \rho_{I_\sigma}(\xi)$ *for all* $\xi \in E_\sigma$); *but nevertheless, L is not an intersection body*. On the other hand, we show that the local equatorial characterization of intersection bodies is possible in *even* dimensions.

We also extend the result of W. Weil [23] to the class of intersection bodies by proving that there is no local characterization of those bodies in odd and even dimensions. We prove that *there exists an origin-symmetric convex C^{∞} body $K \subset \mathbb{R}^n$, $n \geqslant 5$, that is not an intersection body, but has the following property*: *for each $u \in S^{n-1}$ there exists an intersection body I_u centered at the origin and a neighborhood $U_u \subset S^{n-1}$ of u such that the boundaries of K and I_u coincide on U_u*. In odd dimensions this is a consequence of the lack of a local equatorial characterization of intersection bodies mentioned above but we give an independent proof that does not distinguish between even and odd dimensions.

Our proofs for zonoids and intersection bodies are very similar, they are based on almost identical Fourier analytic inversion formulas for the Cosine and Radon transforms. This is one more indication of the remarkable duality between sections and projections (see [15]).

2. Auxiliary results

Our main tool is the Fourier transform of distributions (see [8,9] and [13] for exact definitions and properties) and the connections between the Cosine and the spherical Radon transforms and the Fourier transform.

We start with the connection of the spherical Radon transform and the Fourier transform. A. Koldobsky (see, for example, [13, Lemma 3.7]) proved that

$$\mathfrak{R}g(\xi) = \frac{1}{\pi}\hat{g}(\xi), \quad \forall \xi \in S^{n-1}, \tag{1}$$

provided that g is an even homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$, satisfying $g|_{S^{n-1}} \in L_1(S^{n-1})$.

An immediate consequence of this formula is the following Fourier analytic characterization of intersection bodies (see [13, Theorem 4.1]): *An origin-symmetric star body K is an intersection body if and only if ρ_K, extended to $\mathbb{R}^n$ as a homogenous function of degree -1, represents a positive definite distribution on $\mathbb{R}^n$. When K is infinitely smooth, this is equivalent to $\widehat{\rho_K} \geqslant 0$.*

A very similar connection of the Cosine transform and the Fourier transform was established in [14] (see also [13, p. 155]):

$$Cos\, g(\xi) = -\frac{2}{\pi}\hat{g}(\xi), \quad \forall \xi \in S^{n-1}, \tag{2}$$

provided that g is an even homogeneous function of degree $-n-1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$, satisfying $g|_{S^{n-1}} \in L_1(S^{n-1})$.

As above, one can obtain a very similar Fourier analytic characterization of zonoids (see [13, Theorem 8.6]): *An origin-symmetric star body K is a zonoid if and only if h_K, extended to $\mathbb{R}^n$ as a homogenous function of degree 1, represents a negative definite distribution on $\mathbb{R}^n$. When K is infinitely smooth, this is equivalent to $\widehat{h_K} \leqslant 0$.*

Our next tool is a formula connecting the Fourier transform of powers of the radial function with the derivatives of the parallel section function. Let D be an infinitely smooth origin-symmetric star body in $\mathbb{R}^n$, $\xi \in S^{n-1}$, and let $\xi^\perp = \{x \in \mathbb{R}^n\colon\ x \cdot \xi = 0\}$. We denote by

$$A_{D,\xi}(t) = \mathrm{Vol}_{n-1}\big(D \cap \{\xi^\perp + t\xi\}\big), \quad t \in \mathbb{R},$$

the parallel section function of D in the direction of ξ. The following formula was proved in [7] (see [13, p. 60]):

For any $\xi \in S^{n-1}$ and $k \in \mathbb{N}$, $k \neq n-1$,

$$\widehat{\rho_D^{n-k-1}}(\xi) = (-1)^{k/2}\pi(n-k-1)A_{D,\xi}^{(k)}(0), \tag{3}$$

when k is even, and

$$\widehat{\rho_D^{n-k-1}}(\xi) = (-1)^{\frac{k+1}{2}}2(n-k-1)k! \int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0) - \cdots - A_{D,\xi}^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\,dz, \tag{4}$$

when k is odd.

As a consequence of Eqs. (1), (3), and (4) with $k = n-2$, we obtain the Fourier analytic characterization of intersection bodies (see [13, p. 74] for more details).

Let L be an origin-symmetric star body in $\mathbb{R}^n$ such that ρ_L is infinitely differentiable on S^{n-1}. The body L is an intersection body if and only if $\forall \xi \in S^{n-1}$,

$$(-1)^{(n-2)/2}A_{L,\xi}^{(n-2)}(0) \geqslant 0, \tag{5}$$

when n is even, and

$$(-1)^{(n-1)/2}\int_0^\infty \frac{A_{L,\xi}(z)-A_{L,\xi}(0)-\cdots-A^{(n-3)}_{L,\xi}(0)\frac{z^{n-3}}{(n-3)!}}{z^{n-1}}\,dz\geqslant 0, \tag{6}$$

when n is odd.

Similarly, using the duality relation $h_D=\rho_{D^*}^{-1}$ and Eqs. (2)–(4) with $k=n$, one can obtain the following characterization of zonoids (see [14] or [13, p. 156]):

Let L be an origin-symmetric convex body in $\mathbb{R}^n$ such that h_L is infinitely differentiable on S^{n-1}. The body L is a zonoid (projection body) if and only if $\forall\xi\in S^{n-1}$,

$$(-1)^{n/2}A^{(n)}_{L^*,\xi}(0)\geqslant 0, \tag{7}$$

when n is even, and

$$(-1)^{(n+1)/2}\int_0^\infty \frac{A_{L^*,\xi}(z)-A_{L^*,\xi}(0)-\cdots-A^{(n-1)}_{L^*,\xi}(0)\frac{z^{n-1}}{(n-1)!}}{z^{n+1}}\,dz\geqslant 0, \tag{8}$$

when n is odd.

3. There is no local equatorial characterization of intersection bodies in odd dimensions

To construct a counterexample, it is natural to use (6). This formula shows that one has to use the information about the section function $A_{L,\xi}(z)$ of the body along the whole range of z.

For $0<\varepsilon<1$ and $\xi\in S^{n-1}$, we denote by $U_\varepsilon(\xi)$ the union of caps centered at ξ and $-\xi$:

$$U_\varepsilon(\xi):=\left\{\theta\in S^{n-1}\colon\ |\theta\cdot\xi|\geqslant\sqrt{1-\varepsilon^2}\right\}.$$

We denote by $E_\varepsilon(\xi)$, $0<\varepsilon<1$, the neighborhood of the equator $S^{n-1}\cap\xi^\perp$:

$$E_\varepsilon(\xi):=\left\{\theta\in S^{n-1}\colon\ |\theta\cdot\xi|<\varepsilon\right\}.$$

The following result is crucial for the construction of the counterexample. Its proof is based on the fact that the inversion formula (6) is not local.

Lemma 3.1. *Let $n\geqslant 3$ be odd. Then there exist $\varepsilon>0$ and an absolute constant $c>0$ such that for any $x,\xi\in S^{n-1}$, there exists an even function $f_{x,\xi}$ satisfying $f_{x,\xi}=0$ on $E_\varepsilon(x)$, and $\mathfrak{R}^{-1}f_{x,\xi}\geqslant c$ on $U_\varepsilon(\xi)$.*

Proof. First, we fix $x,\xi\in S^{n-1}$ and find $\varepsilon=\varepsilon(x,\xi)$ and $c=c(x,\xi)$ satisfying the requirement of the lemma. Then we use the compactness argument to produce absolute ε and c.

For fixed $x,\xi\in S^{n-1}$ and some small $\varepsilon>0$ we take two auxiliary infinitely smooth symmetric star bodies M, Q, such that $\rho_M=\rho_Q$ on the closure of $E_\varepsilon(\xi)\cup E_\varepsilon(x)$, and $\rho_M>\rho_Q$ otherwise. We put $f_{x,\xi}=(-1)^{(n-1)/2}(\rho_M-\rho_Q)$. Then $f_{x,\xi}=0$ on $E_\varepsilon(x)$, and $\rho_M=\rho_Q$ on $E_\varepsilon(\xi)$ implies $A^{(k)}_{M,\xi}(0)=A^{(k)}_{Q,\xi}(0)$, $k=0,1,\ldots,n-3$. Thus, (1) and (4) with $k=n-2$ imply

$$\begin{aligned}\mathfrak{R}^{-1} f_{x,\xi}(\xi) &= (-1)^{(n-1)/2}\big(\mathfrak{R}^{-1}\rho_M(\xi) - \mathfrak{R}^{-1}\rho_Q(\xi)\big)\\ &= (-1)^{n-1}(2\pi)^{1-n}(n-2)!\int_0^\infty \frac{A_{M,\xi}(z) - A_{Q,\xi}(z)}{z^{n-1}}\,dz > 0,\end{aligned}$$

since $Q \subseteq M$. We proved that for fixed $x, \xi \in S^{n-1}$ there exist $\varepsilon' = \varepsilon'(x,\xi) > 0$ and $c' = c'(x,\xi)$ such that there exists an even function $f_{x,\xi}$ satisfying $f_{x,\xi} = 0$ on $E_\varepsilon(x)$, and $\mathfrak{R}^{-1} f_{x,\xi}(\xi) \geqslant c'$.

The function $\mathfrak{R}^{-1} f_{x,\xi}$ is continuous on S^{n-1} since M, Q are infinitely smooth (see [13, Lemma 2.4]). Hence, $\mathfrak{R}^{-1} f_{x,\xi} \geqslant c > 0$ on $U_{\varepsilon''}(\xi)$, for some $\varepsilon'' > 0$ and $c = c(x,\xi)$. Put $\tilde\varepsilon = \tilde\varepsilon(x,\xi) = \min(\varepsilon', \varepsilon'')$. We prove that for any x and ξ, there are $\tilde\varepsilon = \tilde\varepsilon(x,\xi) > 0$ and a function $f_{x,\xi}$ such that $f_{x,\xi} = 0$ on $E_{\tilde\varepsilon}(x)$, but $\mathfrak{R}^{-1} f_{x,\xi} \geqslant c$ on $U_{\tilde\varepsilon}(\xi)$, $c = c(x,\xi)$.

Now we use the compactness argument to show that we can choose ε and c independent of x and ξ. We choose a finite set of pairs $\{x_i, \xi_i\}_{i=1}^m$ such that $\{U_{\tilde\varepsilon_i/2}(x_i) \times U_{\tilde\varepsilon_i/2}(\xi_i)\}_{i=1}^m$ cover $S^{n-1} \times S^{n-1}$. We take

$$\varepsilon = \frac{1}{2}\min_{1\leqslant i\leqslant m} \tilde\varepsilon_i \quad \text{and} \quad c = \min_{1\leqslant i\leqslant m} c(x_i, \xi_i).$$

Then, for any (x,ξ), there is a pair (x_i, ξ_i) such that $(x,\xi) \in U_{\tilde\varepsilon_i/2}(x_i) \times U_{\tilde\varepsilon_i/2}(\xi_i)$ and thereby

$$E_\varepsilon(x) \times U_\varepsilon(\xi) \subset E_{\tilde\varepsilon_i}(x_i) \times U_{\tilde\varepsilon_i}(\xi_i).$$

Finally, we may define $f_{x,\xi} = f_{x_i,\xi_i}$. $\square$

Remark 3.2. Note that, dilating M and Q (and thus functions $f_{x,\xi}$), we may assume that c is as large as we want. By the technical reasons that will become clear later, we take $c = 2\mathfrak{R}^{-1}\mathbf{1}$. Moreover, we can assume that the set of functions $\{f_{x,\xi}\}_{x,\xi\in S^{n-1}}$ in the lemma is finite.

Let C_+^∞ be the class of origin-symmetric convex bodies with C^∞ boundary and everywhere positive Gaussian curvature (see [6, p. 25]). The following auxiliary result seems to be well known. It is interesting to note that it is not true without the C_+^∞ assumption though (see [21, pp. 117, 118] and [2,12,16]).

Lemma 3.3. *Let $M \in C_+^\infty$ and let $K(t) = tB_2^n + (1-t)M$ be the Minkowski sum of tB_2^n and $(1-t)M$, $t \in [0,1]$. Then the map $t \to \mathfrak{R}^{-1}\rho_{K(t)}(\xi)$, $\xi \in S^{n-1}$, is continuous.*

Proof. We note first that for any fixed $t \in [0,1]$, the boundary $\partial K(t)$ of $K(t)$ is C^∞. Indeed, $\partial K(t)$ can be parameterized as

$$u \in S^{n-1} \to \nabla h_{(1-t)M}(u) + tu = (1-t)\nabla h_M(u) + tu,$$

where $u \in S^{n-1} \to (1-t)\nabla h_M(u)$ is a parametrization of $(1-t)\partial M$. Here

$$\nabla h_{(1-t)M}(u) = \nu^{-1}(u),$$

and $\nu : (1-t)\partial M \to S^{n-1}$ is the spherical image map (see [6, pp. 22–26] or [21, pp. 103–111]). Since the Gaussian curvatures of M and B_2^n are positive everywhere, one can use the arguments which are similar to those in [21, pp. 106–111], to show that the map $u \in S^{n-1} \to \nabla h_M(u)$ is a C^∞ diffeomorphism. Hence, the map $u \in S^{n-1} \to g_t(u) := (1-t)\nabla h_M(u) + tu$ is also a C^∞ diffeomorphism.

To show that $t \to \mathfrak{R}^{-1}\rho_{K(t)}(\xi)$ is continuous, we pick any $t \in [0,1]$ and take any sequence $\{t_m\}_{m=1}^\infty$ of points from $[0,1]$ converging to t. The map

$$u \in S^{n-1} \to f_t(u) := g_t(u)/\left|g_t(u)\right|$$

is a C^∞ diffeomorphism for any $t \in [0,1]$, and $f_{t_m} \to f_t$ in $C^\infty(S^{n-1})$. Hence, $f_{t_m}^{-1} \to f_t^{-1}$ in $C^\infty(S^{n-1})$. Now, $g_t(f_t^{-1}(\xi)) \in \partial K(t)$ implies $\rho_{K(t)}(\xi) = |g_t(f_t^{-1}(\xi))|$, and $\rho_{K(t_m)}$ converges to $\rho_{K(t)}$ in $C^\infty(S^{n-1})$. Since $\mathfrak{R}$ is a continuous bijection of $C^\infty(S^{n-1})$ to itself, [6, p. 382], the lemma is proved. □

Lemma 3.4. *Let $n \geqslant 5$. For any point $\xi_0 \in S^{n-1}$ there exists $\widetilde{K} \in C_+^\infty$ such that $\mathfrak{R}^{-1}\rho_{\widetilde{K}}(\xi)$ is strictly positive for all $\xi \neq \pm\xi_0$, and $\mathfrak{R}^{-1}\rho_{\widetilde{K}}(\pm\xi_0) = 0$.*

Proof. Fix $n \geqslant 5$. Then there exists $M \in C_+^\infty$ such that $\mathfrak{R}^{-1}\rho_M(\xi)$ is sign-changing (see [13, Lemma 4.10] where an example of such a body is constructed).

For $t \in [0,1]$, consider the Minkowski sum $K(t) = tB_2^n + (1-t)M$. Then $\mathfrak{R}^{-1}\rho_{K(0)}(\xi)$ is sign-changing and there exists $\Lambda' \subset S^{n-1}$ such that $\mathfrak{R}^{-1}\rho_{K(0)}(\xi) < 0$, $\forall \xi \in \Lambda'$. On the other hand, $\mathfrak{R}^{-1}\rho_{K(1)}(\xi) > 0$, $\forall \xi \in S^{n-1}$. By the previous lemma the map $t \to \mathfrak{R}^{-1}\rho_{K(t)}(\xi)$ is continuous, and there is $t_0 \in [0,1]$ such that

$$\mathfrak{R}^{-1}\rho_{K(t_0)}(\xi) \geqslant 0, \quad \forall \xi \in S^{n-1}, \quad \text{and} \quad \mathfrak{R}^{-1}\rho_{K(t_0)}(\xi) = 0, \quad \forall \xi \in \Lambda \subset S^{n-1},$$

for some $\Lambda \neq \emptyset$. Fix any $\xi_0 \in \Lambda$. Consider an even C^∞ smooth function g on S^{n-1} such that

$$g(x) > 0, \quad \forall x \neq \pm\xi_0, \quad \text{and} \quad g(\pm\xi_0) = 0.$$

For $\varepsilon > 0$ define a body $\widetilde{K}$ (depending on ξ_0):

$$\mathfrak{R}^{-1}\rho_{\widetilde{K}}(\xi) = \mathfrak{R}^{-1}\rho_{K(t_0)}(\xi) + \varepsilon g(\xi).$$

Note that $\mathfrak{R}^{-1}\rho_{\widetilde{K}}(\xi)$ is strictly positive for all $\xi \neq \pm\xi_0$, and $\mathfrak{R}^{-1}\rho_{\widetilde{K}}(\pm\xi_0) = 0$. We get

$$\rho_{\widetilde{K}}(x) = \rho_{K(t_0)}(x) + \varepsilon\mathfrak{R}g(x).$$

Since $\mathfrak{R}g$ is a C^∞ function, and $K(t_0) \in C_+^\infty$, we may choose ε small enough so that $\widetilde{K} \in C_+^\infty$. Using the rotation argument, we can take ξ_0 to be arbitrary. □

Theorem 3.5. *Let $n \geqslant 5$ be odd. There exist $\varepsilon > 0$ and a convex symmetric body K that is not an intersection body, but nevertheless $\forall x \in S^{n-1}$ there exists an intersection body L_x such that $\rho_K = \rho_{L_x}$ on $E_\varepsilon(x)$.*

Proof. We define a convex body K and a family of convex bodies $\{L_x\}_{x \in S^{n-1}}$ using $\widetilde{K}$ and functions f_{x,ξ_0} from Lemma 3.1. We fix some small ε satisfying the requirements of Lemma 3.1 and we may assume that $c = 2\mathfrak{R}^{-1}\mathbf{1}$ (see Remark 3.2). Then, define $K = K_{\delta,\xi_0}$ via $\rho_K = \rho_{\widetilde{K}} - \delta$, where for the moment $\delta > 0$ is assumed to be so small that $K \in C_+^\infty$ and $\Re^{-1}\rho_K$ is strictly positive outside $U_\varepsilon(\xi_0)$. Note that $\Re^{-1}\rho_K(\xi_0) < 0$ and thus K is not an intersection body.

Now we define a family of convex bodies $\{L_x\}_{x\in S^{n-1}}$. Since $\widetilde{K} \in C_+^\infty$, we take δ so small that $\rho_{L_x} := \rho_{\widetilde{K}} - \delta + \delta f_{x,\xi_0} > 0$ on S^{n-1} and L_x is convex. Observe that $\rho_{L_x} = \rho_K$ on $E_\varepsilon(x)$ for any $x \in S^{n-1}$.

We can assume that δ is so small that

$$\Re^{-1}\rho_{L_x} = \Re^{-1}\rho_{\widetilde{K}} - \delta\Re^{-1}\mathbf{1} + \delta\Re^{-1} f_{x,\xi_0} > 0$$

on $S^{n-1} \setminus U_\varepsilon(\xi_0)$, since $\Re^{-1}\rho_{\widetilde{K}} > 0$ on $S^{n-1} \setminus U_\varepsilon(\xi_0)$.

To show that bodies L_x are intersection bodies $\forall x \in S^{n-1}$, it is enough to prove that $\Re^{-1}\rho_{L_x} > 0$ on $U_\varepsilon(\xi_0)$. By Remark 3.2, $\min_{x\in S^{n-1}} \Re^{-1} f_{x,\xi_0} \geqslant 2\Re^{-1}\mathbf{1}$ on $U_\varepsilon(\xi_0)$, hence

$$\Re^{-1}\rho_{L_x} = \Re^{-1}\rho_{\widetilde{K}} - \delta\Re^{-1}\mathbf{1} + \delta\Re^{-1} f_{x,\xi_0} \geqslant \delta\Re^{-1}\mathbf{1} > 0$$

on $U_\varepsilon(\xi_0)$. Moreover, $\delta > 0$ can be chosen independently of x since the set of functions $\{f_{x,\xi}\}_{x,\xi\in S^{n-1}}$ in Lemma 3.1 is finite. □

4. There is no local equatorial characterization of zonoids in odd dimensions

The proofs in this section are very similar (in fact, almost identical) to the ones in the previous section.

Lemma 4.1. *Let $n \geqslant 3$ be odd. Then there exist $\varepsilon > 0$ and an absolute constant $c > 0$ such that for any $x, \xi \in S^{n-1}$, there exists an even function $f_{x,\xi}$ satisfying $f_{x,\xi} = 0$ on $E_\varepsilon(x)$, and $Cos^{-1} f_{x,\xi} \geqslant c$ on $U_\varepsilon(\xi)$.*

Proof. The proof follows the same lines as that of Lemma 3.1. One has to change the Spherical Radon transform to the Cosine transform, put support functions instead of radial functions and thus, use section functions of polar bodies together with (2), (4) and (8). □

Remark 4.2. Note that dilating M and Q (and thus functions $f_{x,\xi}$) we may assume that c is as large as we want. For technical reasons, we take $c = 2\,Cos^{-1}\,\mathbf{1}$. Moreover, we can assume that the set of functions $\{f_{x,\xi}\}_{x,\xi\in S^{n-1}}$ in the lemma is finite.

Lemma 4.3. *Let $n \geqslant 3$. For any point $\xi_0 \in S^{n-1}$ there exists a zonoid $\widetilde{K} \in C_+^\infty$ such that $Cos^{-1} h_{\widetilde{K}}(\xi)$ is strictly positive for all $\xi \neq \pm\xi_0$, and $Cos^{-1} h_{\widetilde{K}}(\pm\xi_0) = 0$.*

Proof. Fix $n \geqslant 3$. Then there exists $M \in C_+^\infty$ such that $Cos^{-1} h_M$ is sign-changing (see [13, p. 161], the Fourier Analytic solution of Shephard problem for a construction of a C_+^∞ non-zonoid body).

For $t \in [0, 1]$ consider the Minkowski sum $K(t) = tB_2^n + (1-t)M$. Then $h_{K(t)} = th_{B_2^n} + (1-t)h_M$ is a C^∞-function, $Cos^{-1} h_{K(0)}(\xi)$ is sign-changing and there exists $\Lambda' \subset S^{n-1}$ such that $Cos^{-1} h_{K(0)}(\xi) < 0$, $\forall \xi \in \Lambda'$. On the other hand, $Cos^{-1} h_{K(1)}(\xi) > 0$, $\forall \xi \in S^{n-1}$. The map

$t \to Cos^{-1} h_{K(t)}$ is continuous, since Cos is a continuous bijection of $C^{\infty}(S^{n-1})$ into itself, [6, p. 381]. Hence, there is $t_0 \in [0, 1]$ such that

$$Cos^{-1} h_{K(t_0)} \geqslant 0 \quad \text{and} \quad Cos^{-1} h_{K(t_0)}(\xi) = 0, \quad \forall \xi \in \Lambda \subset S^{n-1},$$

and some $\Lambda \neq \emptyset$. Fix any $\xi_0 \in \Lambda$. Consider an even C^{∞} smooth function g on S^{n-1} such that

$$g(x) > 0, \quad \forall x \neq \pm\xi_0, \quad \text{and} \quad g(\pm\xi_0) = 0.$$

For $\varepsilon > 0$ define a body $\widetilde{K}$:

$$Cos^{-1} h_{\widetilde{K}}(\xi) = Cos^{-1} h_{K(t_0)}(\xi) + \varepsilon g(\xi).$$

Note that $Cos^{-1} h_{\widetilde{K}}(\xi)$ is strictly positive for all $\xi \neq \pm\xi_0$, and $Cos^{-1} h_{\widetilde{K}}(\pm\xi_0) = 0$. Moreover,

$$h_{\widetilde{K}} = h_{K(t_0)} + \varepsilon\, Cos\, g.$$

Since $Cos\, g$ is a continuous function and $K(t_0) \in C_+^{\infty}$, we may choose ε small enough so that $\widetilde{K} \in C_+^{\infty}$. Using the rotation argument, we can take ξ_0 to be arbitrary. □

Theorem 4.4. *Let $n \geqslant 3$ be odd. There exist $\varepsilon > 0$ and a convex body K that is not a zonoid, but nevertheless $\forall x \in S^{n-1}$ there exists a zonoid L_x such that $h_K = h_{L_x}$ on $E_\varepsilon(x)$.*

Proof. We define a convex body K and a family of convex bodies $\{L_x\}_{x \in S^{n-1}}$ using the zonoid $\widetilde{K}$ and functions f_{x,ξ_0} from Lemma 4.1. We fix some small ε satisfying the requirements of Lemma 4.1 with $c = 2\, Cos^{-1} \mathbf{1}$ (see Remark 4.2). Then, define $K = K_{\delta,\xi_0}$ via $h_K = h_{\widetilde{K}} - \delta$, where for the moment $\delta > 0$ is assumed to be so small that $K \in C_+^{\infty}$ and $Cos^{-1} h_K$ is strictly positive outside $U_\varepsilon(\xi_0)$. Note that $Cos^{-1} h_K(\xi_0) < 0$ and thus K is not a zonoid.

Now we define a family of convex bodies $\{L_x\}_{x \in S^{n-1}}$. Since $\widetilde{K} \in C_+^{\infty}$, we take δ so small that $h_{L_x} := h_{\widetilde{K}} - \delta + \delta f_{x,\xi_0} > 0$ on S^{n-1} and L_x is convex. Observe that $h_{L_x} = h_K$ on $E_\varepsilon(x)$ for any $x \in S^{n-1}$.

We can assume that δ is so small that

$$Cos^{-1} h_{L_x} = Cos^{-1} h_{\widetilde{K}} - \delta\, Cos^{-1} \mathbf{1} + \delta\, Cos^{-1} f_{x,\xi_0} > 0$$

on $S^{n-1} \setminus U_\varepsilon(\xi_0)$, since $Cos^{-1} h_{\widetilde{K}} > 0$ on $S^{n-1} \setminus U_\varepsilon(\xi_0)$.

To show that bodies L_x are zonoids $\forall x \in S^{n-1}$, it is enough to prove that $Cos^{-1} h_{L_x} > 0$ on $U_\varepsilon(\xi_0)$. By Remark 4.2, $\min_{x \in S^{n-1}} Cos^{-1} f_{x,\xi_0} > 2\, Cos^{-1} \mathbf{1}$ on $U_\varepsilon(\xi_0)$, hence

$$Cos^{-1} h_{L_x} = Cos^{-1} h_{\widetilde{K}} - \delta\, Cos^{-1} \mathbf{1} + \delta\, Cos^{-1} f_{x,\xi_0} \geqslant \delta\, Cos^{-1} \mathbf{1} > 0$$

on $U_\varepsilon(\xi_0)$, and the result follows. □

5. There is a local equatorial characterization of intersection bodies and zonoids in even dimensions

We consider at first intersection bodies. The proof of the following lemma is obtained by a straightforward repetition of the argument from [13, p. 60], and we omit the details.

Lemma 5.1. *Let $g(x)$ be an even homogeneous function of degree -1 such that $g(x)$ is nonnegative and infinitely smooth on S^{n-1}. Then*

$$\hat{g}(\xi) = (-1)^{(n-2)/2} \pi A_{g,\xi}^{(n-2)}(0),$$

where

$$A_{g,\xi}(z) = \int\limits_{\{y \in \mathbb{R}^n:\ y \cdot \xi = z\}} \chi_{[0,1]}\bigl(1/g(y)\bigr)\, dy, \quad \xi \in S^{n-1}.$$

Theorem 5.2. *Let n be even and let $K \subset \mathbb{R}^n$ be an origin-symmetric convex body. Assume that for any great sphere $\xi^\perp \cap S^{n-1}$, there exist an intersection body L_ξ and a neighborhood $E_{\varepsilon(\xi)}(\xi)$ of $\xi^\perp \cap S^{n-1}$ such that the radial functions of K and L_ξ coincide at all points of $E_{\varepsilon(\xi)}(\xi)$; then K is an intersection body.*

Proof. If K and L_ξ are infinitely smooth, then it is enough to observe that $\rho_K(u) = \rho_{L_\xi}(u)$, $\forall u \in E_{\varepsilon(\xi)}(\xi)$ implies $A_{K,\xi}(t) = A_{L_\xi,\xi}(t)$ for sufficiently small t and apply (5).

Consider the general case. It was proved by A. Koldobsky that an origin-symmetric body K is an intersection body if and only if ρ_K represents a positive definite distribution (see, for example, Theorem 4.1 in [13]). Thus, it is enough to show that

$$\langle \widehat{\rho_K}, \varphi \rangle \geqslant 0, \quad \text{for all nonnegative test functions } \varphi \text{ on } \mathbb{R}^n.$$

Using the definition of the Fourier Transform of distributions (see [13, Section 2.5]), and passing to the polar coordinates, we get

$$\langle \widehat{\rho_K}, \varphi \rangle = \langle \rho_K, \hat{\varphi} \rangle = \int\limits_{\mathbb{R}^n} \rho_K(x) \hat{\varphi}(x)\, dx = \int\limits_{S^{n-1}} \rho_K(\theta) \int\limits_0^\infty r^{n-2} \hat{\varphi}(r\theta)\, dr\, d\theta.$$

Observe that the function $\alpha(x) := \int_0^\infty r^{n-2} \hat{\varphi}(rx)\, dr$, $x \in R^{n-1} \setminus \{0\}$, is homogeneous of degree $-n+1$ and infinitely smooth. Hence, we may apply equality (4.3), p. 72, together with Lemma 3.7, p. 53, from [13], to claim that there exists an infinitely smooth *nonnegative* homogeneous of degree -1 function

$$g(x) = \frac{1}{2} \int\limits_{\mathbb{R}} \varphi(tx)\, dt \quad \text{such that} \quad \hat{g}(\theta) = \alpha(\theta), \quad \forall \theta \in S^{n-1}.$$

Thus,

$$\int_{S^{n-1}} \rho_K(\theta) \int_0^\infty r^{n-2}\hat{\varphi}(r\theta)\,dr\,d\theta = \int_{S^{n-1}} \rho_K(\theta)\hat{g}(\theta)\,d\theta.$$

Using a partition of unity on S^{n-1}, we can write

$$g(\theta) = \sum_{j=1}^m g_j(\theta) = \sum_{j=1}^m \frac{1}{2}\int_{\mathbb{R}} \varphi_j(t\theta)\,dt, \quad \theta \in S^{n-1},$$

where $\operatorname{supp} g_j|_{S^{n-1}} \subset U_{\varepsilon_j}(\xi_j)$ are small enough.

By the previous lemma, $\operatorname{supp} g_j|_{S^{n-1}} \subset U_{\varepsilon_j}(\xi_j)$ implies $\operatorname{supp} \widehat{g_j}|_{S^{n-1}} \subset E_{\varepsilon_j}(\xi_j)$. Hence,

$$\langle \widehat{\rho_K}, \varphi\rangle = \sum_{j=1}^m \int_{S^{n-1}} \rho_K(\theta)\widehat{g_j}(\theta)\,d\theta = \sum_{j=1}^m \int_{E_{\varepsilon_j}(\xi_j)} \rho_K(\theta)\widehat{g_j}(\theta)\,d\theta = \sum_{j=1}^m \int_{E_{\varepsilon_j}(\xi_j)} \rho_{L_{\xi_j}}(\theta)\widehat{g_j}(\theta)\,d\theta$$

$$= \sum_{j=1}^m \int_{S^{n-1}} \rho_{L_{\xi_j}}(\theta)\widehat{g_j}(\theta)\,d\theta = \sum_{j=1}^m \langle \widehat{\rho_{L_{\xi_j}}}, \varphi_j\rangle \geqslant 0. \qquad \square$$

The following result was obtained independently by G. Panina [18] and P. Goodey and W. Weil [10]. Its proof could be also obtained by the arguments similar to those in the previous proof, and we omit it.

Theorem 5.3. *Let n be even and let $K \subset \mathbb{R}^n$ be an origin-symmetric convex body. Assume that for any great sphere $\xi^\perp \cap S^{n-1}$, there exists a zonoid Z_ξ and a neighborhood $E_{\varepsilon(\xi)}(\xi)$ of $\xi^\perp \cap S^{n-1}$ such that the boundaries of K and Z_ξ coincide at all points where the exterior unit vector belongs to $E_{\varepsilon(\xi)}(\xi)$; then K is a zonoid.*

6. There is no local characterization of intersection bodies

In this section we prove the analog of the result of W. Weil [23] for zonoids. Our proof is different from the one of W. Weil. We show that, given $x, \xi \in S^{n-1}$, one can construct a function f which is zero around x, but such that the inverse spherical Radon transform of f is positive around ξ. For convenience of the reader we split the proof of this auxiliary result (see Lemma 6.4) into four statements. We will use the following notation

$$\Im_{\varepsilon,x} = \bigl\{f \in C^\infty\bigl(S^{n-1}\bigr)\colon\ f = 0 \text{ on } U_\varepsilon(x)\bigr\}, \quad 0 < \varepsilon < 1.$$

Lemma 6.1. *Let $n \geqslant 3$, and let $\xi, x \in S^{n-1}$ be two orthogonal vectors. Assume that any $f \in \Im_{1/4,x}$ satisfies $\Re^{-1} f(\xi) = 0$. Then for any pair of orthogonal vectors $u, v \in S^{n-1}$ we have $f \in \Im_{1/4,u}$ implies $\Re^{-1} f(v) = 0$.*

Proof. For any two pairs of orthogonal unit vectors (ξ, x), (u, v) there exists a rotation $\rho \in SO(n)$ satisfying $u = \rho(x)$, $v = \rho(\xi)$. Since $\mathfrak{R}^{-1}$ commutes with rotations, the result follows. $\square$

Lemma 6.2. *Let $n \geqslant 3$, and let $\xi \in x^{\perp}$. Assume that any $f \in \Im_{1/4,x}$ satisfies $\mathfrak{R}^{-1} f(\xi) = 0$. Then $\mathfrak{R}^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,\xi}$.*

Proof. Take any $u \in U_{1/4}(\xi)$. Let $\rho \in SO(n)$, $\rho(\xi) = u$, where ξ is rotated into u inside $U_{1/4}(\xi)$ in the plane containing ξ, u and the origin. Then $\rho(x) \in U_{1/4}(x)$, and $\Im_{1/2,x} \subset \Im_{1/4,\rho(x)}$. Moreover, $\mathfrak{R}^{-1} f(u) = 0$ since $\mathfrak{R}^{-1}$ commutes with rotations. The point u was chosen arbitrarily in $U_{1/4}(\xi)$, hence $\mathfrak{R}^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,\xi}$. $\square$

Lemma 6.3. *Let $n \geqslant 3$, and let $\xi \in x^{\perp}$. Then there exists a function $f = f_{x,\xi}$ on S^{n-1} satisfying $f_{x,\xi} = 0$ on $U_{1/4}(x)$, but $\mathfrak{R}^{-1} f_{x,\xi}(\xi) \neq 0$.*

Proof. Assume the contrary. Then $\mathfrak{R}^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,\xi}$ by Lemma 6.2. Take any vector $y \in S^{n-1}$, and find a vector $q \in x^{\perp} \cap y^{\perp}$. Let $\rho \in SO(n)$ be such that $\rho(x) = x$, $\rho(\xi) = q$. Observe that $f \in \Im_{\epsilon,x}$ implies $f(\rho(\cdot)) \in \Im_{\epsilon,x}$. Since $\mathfrak{R}^{-1}$ commutes with rotations, $\mathfrak{R}^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,\xi}$ yields $\mathfrak{R}^{-1}(\Im_{1/2,x}) \subset \Im_{1/4,q}$. Take two pairs of orthogonal vectors (x, q) and (q, y). By Lemma 6.1, we have $\mathfrak{R}^{-2} f(y) = 0$. Thus, $\mathfrak{R}^{-2} f \equiv 0$, a contradiction. $\square$

Lemma 6.4. *Let $n \geqslant 3$. Then there exist $\varepsilon > 0$ and an absolute constant $c > 0$ such that for any $x, \xi \in S^{n-1}$, there exists an even function $f_{x,\xi}$ satisfying $f_{x,\xi} = 0$ on $U_\varepsilon(x)$, and $\mathfrak{R}^{-1} f_{x,\xi} \geqslant c$ on $U_\varepsilon(\xi)$.*

Proof. We fix points x and ξ, and provide $\varepsilon > 0$ and $c > 0$ depending on x, ξ such that there is a function $f_{x,\xi}$ satisfying $f_{x,\xi} = 0$ on $U_\varepsilon(x)$, and $\mathfrak{R}^{-1} f_{x,\xi} \geqslant c > 0$ on $U_\varepsilon(\xi)$. Then we use the compactness argument to prove the statement of the lemma.

Let $\xi \notin x^{\perp}$. Then there exists $\varepsilon > 0$, such that $\xi \notin E_\varepsilon(x)$. For any function g the values of $\mathfrak{R} g$ on $U_\varepsilon(x)$ depend only on the values of g on $E_\varepsilon(x)$. Hence, we may consider an even C^∞-function g such that $g(\pm\xi) > 0$ and $g(v) = 0$, for $v \in E_\varepsilon(x)$ and define $f_{x,\xi} = \mathfrak{R} g(x)$.

Let $\xi \in x^{\perp}$. Then, the previous lemma implies the existence of $\varepsilon = \varepsilon(x, \xi) = 1/8$, and a function $f = f_{x,\xi}$ on S^{n-1} satisfying $f_{x,\xi} = 0$ on $U_\varepsilon(x)$, but $\mathfrak{R}^{-1} f_{x,\xi}(\xi) > 0$ (change the sign of $f_{x,\xi}$ if necessary).

Thus, we proved that for any x and ξ, there is $\varepsilon' = \varepsilon'(x, \xi) > 0$ and there is a function $f_{x,\xi}$ such that $f_{x,\xi} = 0$ on $U_{\varepsilon'}(x)$, but $\mathfrak{R}^{-1} f_{x,\xi}(\pm\xi) \geqslant c'$, $c' = c'(x, \xi) > 0$. From the continuity of the function $\mathfrak{R}^{-1} f_{x,\xi}$ we get that $\mathfrak{R}^{-1} f_{x,\xi} \geqslant c$, $c = c(x, \xi) > 0$ on $U_{\varepsilon''}(\xi)$, for some $\varepsilon'' > 0$. Take $\tilde{\varepsilon} = \tilde{\varepsilon}(x, \xi) = \min(\varepsilon', \varepsilon'')$. We show that for any x and ξ, there is $\tilde{\varepsilon} = \tilde{\varepsilon}(x, \xi) > 0$ and there is a function $f_{x,\xi}$ such that $f_{x,\xi} = 0$ on $U_{\tilde{\varepsilon}}(x)$, but $\mathfrak{R}^{-1} f_{x,\xi} \geqslant c$ on $U_{\tilde{\varepsilon}}(\xi)$, $c = c(x, \xi) > 0$.

Now we use the compactness argument to prove that we can choose an ε and c independent of x and ξ. We choose a finite set of $\{x_i, \xi_i\}_{i=1}^m$ such that $\{U_{\tilde{\varepsilon}_i/2}(x_i) \times U_{\tilde{\varepsilon}_i/2}(\xi_i)\}_{i=1}^m$ covers $S^{n-1} \times S^{n-1}$. We take

$$\varepsilon = \frac{1}{2} \min_{1 \leqslant i \leqslant m} \tilde{\varepsilon}_i \quad \text{and} \quad c = \min_{1 \leqslant i \leqslant m} c(x_i, \xi_i).$$

Then for any (x, ξ) there is a (x_i, ξ_i) such that

$$U_\varepsilon(x) \times U_\varepsilon(\xi) \subset U_{\tilde{\varepsilon}_i}(x_i) \times U_{\tilde{\varepsilon}_i}(\xi_i),$$

and we may define $f_{x,\xi} = f_{x_i,\xi_i}$. □

Theorem 6.5. *Let $n \geqslant 5$. There exists a convex body K that is not an intersection body, such that $\forall x \in S^{n-1}$ there exists an $\varepsilon(x)$ and an intersection body L_x such that $\rho_K = \rho_{L_x}$ on $U_{\varepsilon(x)}(x)$.*

Proof. Repeat the proof of Theorem 1. □

Acknowledgments

The authors are very grateful to Paul Goodey, Alexander Koldobsky, Jeffrey Schlaerth and Wolfgang Weil for many useful discussions.

References

[1] E.D. Bolker, A class of convex bodies, Trans. Amer. Math. Soc. 145 (1969) 323–345.
[2] J. Boman, The sum of two plane convex C^∞ sets is not always C^5, Math. Scand. 66 (2) (1990) 216–224.
[3] J. Bourgain, J. Lindenstrauss, Projection bodies, in: Israel Seminar (G.A.F.A.), 1986–1987, in: Lecture Notes in Math., vol. 1317, Springer-Verlag, Berlin, 1988, pp. 250–269.
[4] J. Bourgain, J. Lindenstrauss, V. Milman, Approximation of zonoids by zonotopes, Acta Math. 162 (1989) 73–141.
[5] R.J. Gardner, Intersection bodies and the Busemann–Petty problem, Trans. Amer. Math. Soc. 342 (1994) 435–445.
[6] R.J. Gardner, Geometric Tomography, Cambridge Univ. Press, New York, 1995.
[7] R.J. Gardner, A. Koldobsky, Th. Schlumprecht, An analytic solution to the Busemann–Petty problem on sections of convex bodies, Ann. of Math. 149 (1999) 691–703.
[8] I.M. Gelfand, G.E. Shilov, Generalized Functions 1, Academic Press, New York, 1964.
[9] I.M. Gelfand, N.Ya. Vilenkin, Generalized Functions 4. Applications of Harmonic Analysis, Academic Press, New York, 1964.
[10] P. Goodey, W. Weil, Centrally symmetric bodies and the spherical Radon transform, J. Differential Geom. 35 (1992) 675–688.
[11] P. Goodey, W. Weil, Zonoids and generalizations, in: P.M. Gruber, J.M. Wills (Eds.), Handbook of Convex Geometry, North-Holland, Amsterdam, 1993, pp. 1297–1326.
[12] C.O. Kiselman, Smoothness of vector sums of plane convex sets, Math. Scand. 60 (2) (1987) 239–252.
[13] A. Koldobsky, Fourier Analysis in Convex Geometry, Math. Surveys Monogr., Amer. Math. Soc., Providence, RI, 2005.
[14] A. Koldobsky, D. Ryabogin, A. Zvavitch, Projections of convex bodies and the Fourier transform, Israel J. Math. 139 (2004) 361–380.
[15] A. Koldobsky, D. Ryabogin, A. Zvavitch, Fourier analytic methods in the study of sections and projections of convex bodies, in: L. Brandolini, L. Cozani, A. Iosevich, G. Travaglini (Eds.), Proceedings of Workshop on Convexity and Fourier Analysis, Birkhäuser, 2004, pp. 119–131.
[16] S.G. Krantz, H.R. Parks, On the vector sum of two convex sets in space, Canad. J. Math. 43 (2) (1991) 347–355.
[17] E. Lutwak, Intersection bodies and dual mixed volumes, Adv. Math. 71 (1988) 232–261.
[18] G.Yu. Panina, The representation of an n-dimensional body in the form of a sum of $(n-1)$-dimensional bodies, Soviet J. Contemporary Math. Anal. 23 (1988) 91–103 (English transl.).
[19] C.M. Petty, Projection bodies, in: Proc. Coll. Convexity, Copenhagen, 1965, Kobenhavns Univ. Mat. Inst., 1965, pp. 234–241.
[20] R. Schneider, Zu einem problem von Shephard über die projektionen konvexer Körper, Math. Z. 101 (1967) 71–82.
[21] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge University Press, Cambridge, 1993.
[22] R. Schneider, W. Weil, Zonoids and related topics, in: P.M. Gruber, J.M. Wills (Eds.), Convexity and Its Applications, Birkhäuser, Basel, 1983, pp. 296–317.
[23] W. Weil, Blaschkes Problem der lokalen Charakterisierung von Zonoiden, Arch. Math. 29 (1977) 655–659.
[24] G. Zhang, Centered bodies and dual mixed volumes, Trans. Amer. Math. Soc. 345 (1994) 777–801.